LNCT GROUP OF COLLEGES

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Subject: Control System (EX-405)

Unit: V

Topic: Design of Control Systems

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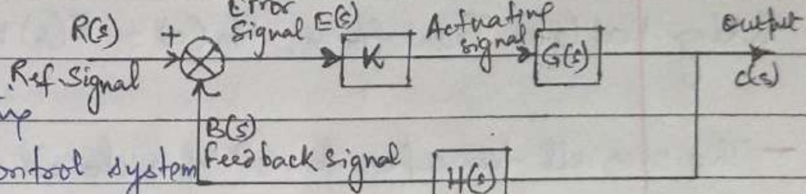
CONTROLLERS AND COMPENSATING NETWORKS

- An automatic control system is used to maintain its output within desirable limits by means of a control action. The control action may operate through either mechanical, hydraulic, pneumatic or electro-mechanical means i.e. Controllers can be electrical, hydraulic, pneumatic, electro-mechanical or electronic types.

Control Actions-

1) Proportional Control-



- In proportional control the actuating signal for the control action in a control system is proportional to the error signal.

- The error signal being the difference between the reference input signal and the feedback signal obtained from the output.

From the diagram $C(s) = K\,G(s)$ or $K = \dfrac{C(s)}{G(s)}$

- It is desirable that the control system be underdamped for quick response because an underdamped control system exhibits exponentially decaying osillations in the output time responce.

- The sluggish over damped response of a control system can be made faster by increasing forward path gain of the system. The increase in forward path gain reduces the steady state error, but at the same time max^{m} overshoot is increased.

For satisfactory performance of a control system a convenient adjustment has to be made b/w the max^{m} overshoot and steady state error.

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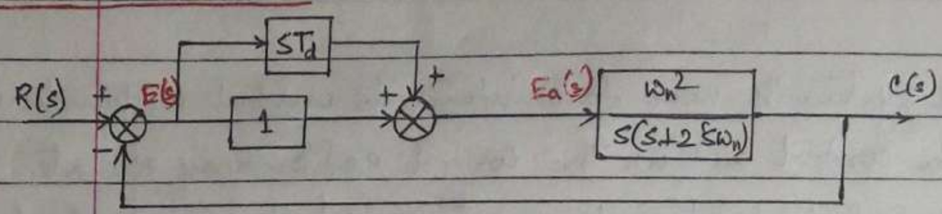
2) Derivative Control-



- For the derivative control action the actuating signal consists of proportional error signal added with the derivative of the error signal. The actuating signal is given by-

$$e_a(t) = e(t) + T_d \frac{de(t)}{dt} \qquad \text{where } T_d = \text{a Constant}$$

Taking Laplace Transform, $E_a(s) = E(s) + s.T_d E(s) = (1+sT_d)\,E(s)$

- The overall transfer func^n of a closed loop 2nd order control system using derivative control is given by-

$$\frac{C(s)}{R(s)} = (1+sT_d)\cdot\frac{\omega_n^2}{s(s+2\xi\omega_n)} \Big/ 1+(1+sT_d)\cdot\frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

$$\frac{C(s)}{R(s)} = \frac{(1+sT_d)\,\omega_n^2}{s^2+(2\xi\omega_n+\omega_n^2T_d)\,s+\omega_n^2} \quad \text{—(1)}$$

- Characteristic eq^n for the overall transfer function is given by-

$$s^2+(2\xi\omega_n+\omega_n^2T_d)\,s+\omega_n^2 = 0 \quad \text{—— (2)}$$

- The damping ratio of the system using derivative control is -

$$\xi' = \frac{2\xi\omega_n+\omega_n^2T_d}{2\omega_n}$$

or $$\boxed{\xi' = \xi + \frac{\omega_nT_d}{2}} \quad \text{—— (3)}$$

- Thus using derivative control the effective damping is increased and therefore max^m overshoot is reduced.

Rewriting the overall transfer function given by eq^n (1) -

$$\frac{C(s)}{R(s)} = \frac{T_d.\omega_n^2\left[s+1/T_d\right]}{s^2+2\omega_n\left[\xi+\frac{\omega_nT_d}{2}\right]s+\omega_n^2} = \omega_n^2T_d\cdot\frac{(s+1/T_d)}{s^2+2\xi'\omega_ns+\omega_n^2}$$

- from the block diagram, the forward path transfer function is -

$$G(s) = (1+sT_d) \cdot \frac{\omega_n^2}{s(s+2\xi\omega_n)} \quad \text{--- (4)}$$

and feed back path transfer function is $H(s) = 1$

- The transfer function relating $E(s)$ and $R(s)$ is given by -

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\frac{(1+sT_d)\cdot\omega_n^2}{s(s+2\xi\omega_n)}\cdot 1}$$

$$\frac{E(s)}{R(s)} = \frac{s(s+2\xi\omega_n)}{s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2} \quad \text{--- (5)}$$

- for a unit ramp input, $R(s) = 1/s^2$

$$E(s) = \frac{1}{s^2} \cdot \frac{s(s+2\xi\omega_n)}{s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2} \quad \text{--- (6)}$$

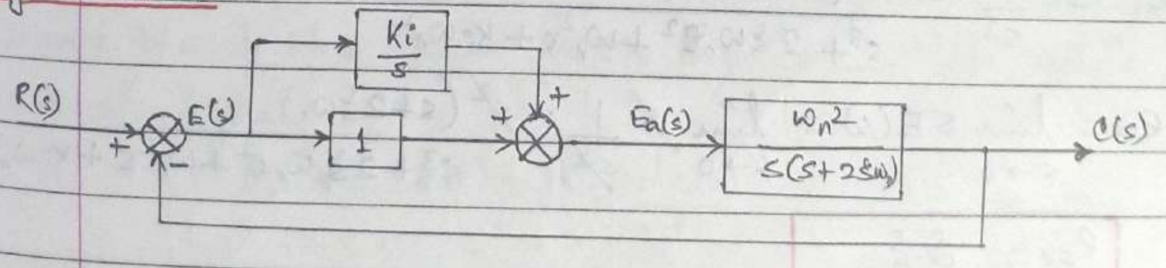
- Steady state error for unit ramp input is -

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s}\cdot\frac{1}{\cancel{s^2}}\cdot\frac{\cancel{s}(s+2\xi\omega_n)}{[s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2]}$$

or $\boxed{e_{ss} = \frac{2\xi}{\omega_n}}$ --- (7)

- Using derivative control, the natural freq. ω_n is unchanged but a zero $(s = -1/T_d)$ is added resulting in decrease in rise time (t_r)

3) Integral Control -



- for integral control action the actuating signal consists of proportional error signal added with integral of the error signal, which is given by -

$$e_a(t) = e(t) + K_i\int e(t)\,dt \quad \text{--- (1)} \qquad K_i = \text{Constant}$$

The Laplace transform of the actuating signal incorporating integral control is-

$$E_a(s) = E(s) + K_i \frac{E(s)}{s} \quad \text{— (2)}$$

- from the block diagram, the T.F. of a closed loop 2nd order control system using integral control is -

$$\frac{C(s)}{R(s)} = \left(1+ K_i/s\right)\left[\frac{\omega_n^2}{s(s+2\xi\omega_n)}\right] \Big/ 1+\left(1+\frac{K_i}{s}\right)\left[\frac{\omega_n^2}{s(s+2\xi\omega_n)}\right]\times 1$$

$$\frac{C(s)}{R(s)} = \frac{(s+K_i)\omega_n^2}{s^3+2\xi\omega_n s^2+\omega_n^2 s+K_i\omega_n^2} \quad \text{— (3)}$$

- characteristic Eqn for the overall T.F. is -

$$s^3+2\xi\omega_n s^2+\omega_n^2 s+K_i\omega_n^2 = 0$$

* If $2\xi\omega_n < K_i$ - two out of three roots have +ve real part
* If $2\xi\omega_n > K_i$ - all three roots of the C.E. have -ve real parts.

- from the block diagram; $G(s) = \dfrac{(s+K_i)\omega_n^2}{s^2(s+2\xi\omega_n)}$ — (4)

and feedback path T.F. is $H(s) = 1$

Now $$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\dfrac{(s+K_i)\omega_n^2}{s^2(s+2\xi\omega_n)}\cdot 1} = \frac{s^2(s+2\xi\omega_n)}{s^3+2\xi\omega_n s^2+\omega_n^2 s+K_i\omega_n^2} \quad \text{—(5)}$$

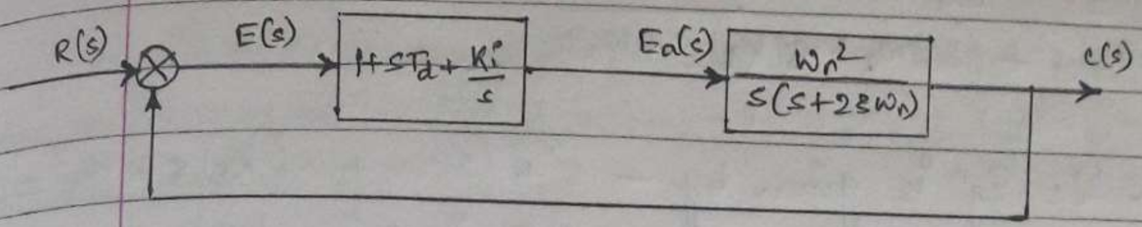
- for unit ramp input with integral control the steady state error e_{ss} is zero. Now for a unit parabolic input i.e. $R(s) = 1/s^3$ the steady state error is given by-

$$E(s) = \frac{1}{s^3}\cdot\frac{s^2(s+2\xi\omega_n)}{s^3+2\xi\omega_n s^2+\omega_n^2 s+K_i\omega_n^2}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s}\cdot\frac{1}{\cancel{s^3}}\cdot\frac{\cancel{s^2}(s+2\xi\omega_n)}{s^3+2\xi\omega_n s^2+\omega_n^2 s+K_i\omega_n^2}$$

$$\boxed{e_{ss} = \frac{2\xi}{K_i\omega_n}}$$

4) Proportional Plus Derivative Plus Integral Control (PID Control) -



- For PID control the actuating signal consists of proportional error signal added with derivative and integral of the error signal.

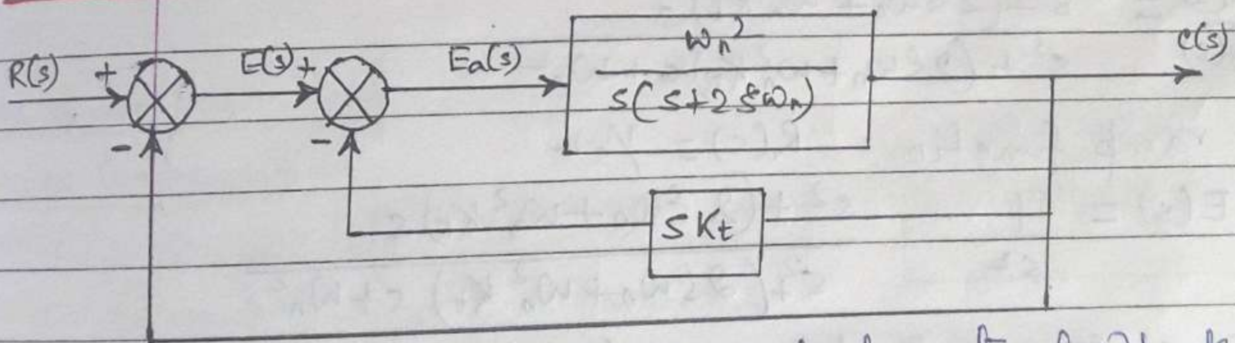
$$e_a(t) = e(t) + T_d \frac{de(t)}{dt} + K_i \int e(t)\, dt$$

The Laplace Transform of the actuating signal is given by -

$$E_a(s) = E(s) + sT_d E(s) + \frac{K_i}{s} E(s)$$

or $$E_a(s) = E(s)\left[1 + sT_d + \frac{K_i}{s}\right]$$

5) Derivative Feedback Control -



- It is also known as rate feedback or tachometer feedback control. For the derivative feedback control the actuating signal is obtained as the difference b/w proportional error signal and derivative (rate) of the output signal.

$$e_a(t) = e(t) - K_t \frac{dc(t)}{dt} \qquad K_t = \text{Constant}.$$

- The Laplace transform of the actuating signal is given by -

$$E_a(s) = E(s) - sK_t\, C(s)$$

- From the block diagram, the transfer func^n is given by-

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

- characteristic Eq^n is given by - $s^2 + (2\xi\omega_n + \omega_n^2 K_t)s + \omega_n^2 = 0$
- The damping ratio for the above C.E. is given by-

$$\xi' = \frac{2\xi\omega_n + \omega_n^2 K_t}{2\omega_n}$$

$$\boxed{\xi' = \xi + \frac{\omega_n K_t}{2}}$$

- The damping ratio is increased by using derivative feedback control and thus max^m overshoot is reduced, but rise time is increased.

- Now $\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \frac{\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s} \cdot 1}$

$$\frac{E(s)}{R(s)} = \frac{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s}{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

for a unit ramp function, $R(s) = 1/s^2$

$$\therefore E(s) = \frac{1}{s^2} \cdot \frac{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s}{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

- The steady state error is determined as-

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$e_{ss} = \lim_{s \to 0} \cancel{s} \cdot \frac{1}{s^{\cancel{2}}} \cdot \frac{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s}{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{\cancel{s}} \cdot \frac{\cancel{s}\left[s + (2\xi\omega_n + \omega_n^2 K_t)\right]}{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

$$\boxed{e_{ss} = \frac{2\xi}{\omega_n} + K_t}$$

COMPENSATION TECHNIQUES

- Sometimes it is necessary to compensate an unstable system to make it stable or it may be necessary to improve the existing system to satisfy or meet the required specifications.

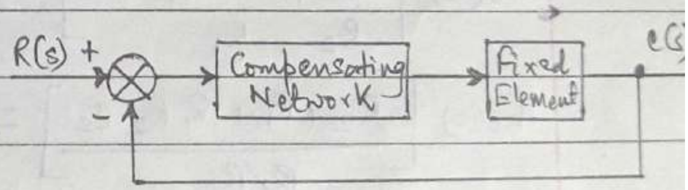
- The compensation may be in the form of adjustment of forward path gain or inserting a compensating device in control systems. Main requirement of the control system is accuracy and stability.

- For greater accuracy, steady state error should be small, but to reduce steady state error, gain of the amplifier must be increased, overshoot will also increase and stability will decrease.

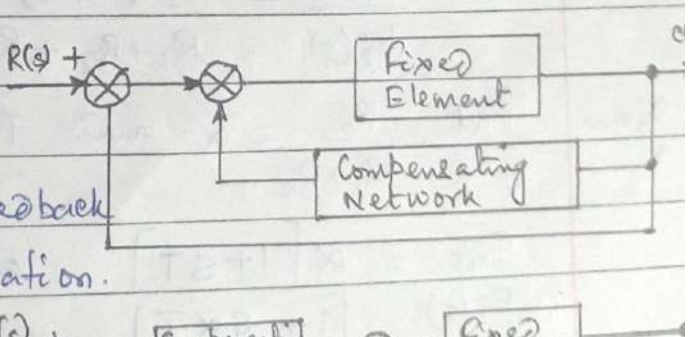
- But interest lies in both accuracy and stability, which can be obtained by connecting a compensating circuit b/w error detector and plant known as compensation.

Types of Compensation-

1) Series Compensation-

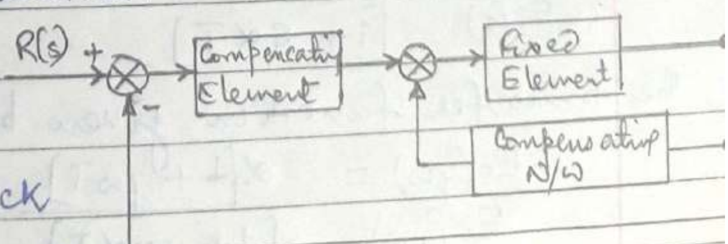


- When a compensating N/w is inserted in the forward path, it is known as series or cascade compensation.



2) Feedback Compensation-

- When a compensator is inserted in feedback path, this is called feedback compensation.

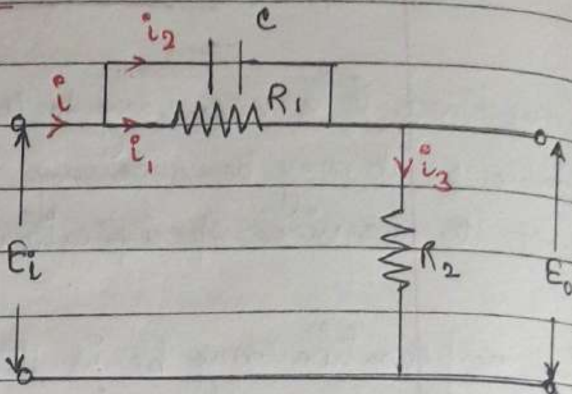


3) Load Compensation-

- A combination of series and feedback compensation or combined cascade and feedback compensation is called load compensation.

Design of Compensation using Bode's Plot –

1) Phase-Lead Compensation –



From the circuit diagram –

$$i = i_1 + i_2$$

where $i_1 = \dfrac{E_i - E_0}{R_1}$

$$i_2 = C\frac{d}{dt}(E_i - E_0)$$

$$\therefore \quad i = \frac{E_i - E_0}{R_1} + C\frac{d}{dt}(E_i - E_0) \quad \text{— (1)}$$

and $\quad i_3 = i = \dfrac{E_0}{R_2} \quad$ — (2)

Put (2) to (1) $\rightarrow$ $\dfrac{E_0}{R_2} = \dfrac{E_i - E_0}{R_1} + C\dfrac{d}{dt}(E_i - E_0)$ — (3)

Take L.T. of (3) $\rightarrow$ $\dfrac{E_0(s)}{R_2} = \dfrac{1}{R_1}\left[E_i(s) - E_0(s)\right] + Cs\left[E_i(s) - E_0(s)\right]$

$$\frac{E_0(s)}{R_2} + \frac{E_0(s)}{R_1} + Cs\,E_0(s) = \frac{1}{R_1}E_i(s) + Cs\,E_i(s)$$

$$E_0(s)\left[\frac{1}{R_2} + \frac{1}{R_1} + Cs\right] = E_i(s)\left[\frac{1}{R_1} + Cs\right]$$

$$E_0(s)\left[\frac{R_2 + R_1 + R_1R_2Cs}{\cancel{R_1}R_2}\right] = E_i(s)\left[\frac{1 + R_1Cs}{\cancel{R_1}}\right]$$

or $\quad \dfrac{E_0(s)}{E_i(s)} = \dfrac{R_2 + R_1R_2Cs}{R_1 + R_2 + R_1R_2Cs} = \dfrac{R_2\left[1 + R_1Cs\right]}{(R_1 + R_2)\left[1 + \dfrac{R_1 R_2 Cs}{R_1 + R_2}\right]}$ — (4)

Now Put $\dfrac{R_2}{R_1 + R_2} = \alpha$ and $T = R_1C$

$$\frac{E_0(s)}{E_i(s)} = \frac{\alpha\left[1 + sT\right]}{\left[1 + s\alpha T\right]} \quad \text{and} \quad \alpha < 1 \quad \text{— (5)}$$

The transfer function given by eqⁿ(5) in sinusoidal form is

$$\frac{E_0(j\omega)}{E_i(j\omega)} = \frac{\alpha\left[1 + j\omega T\right]}{\left[1 + j\omega\alpha T\right]}$$

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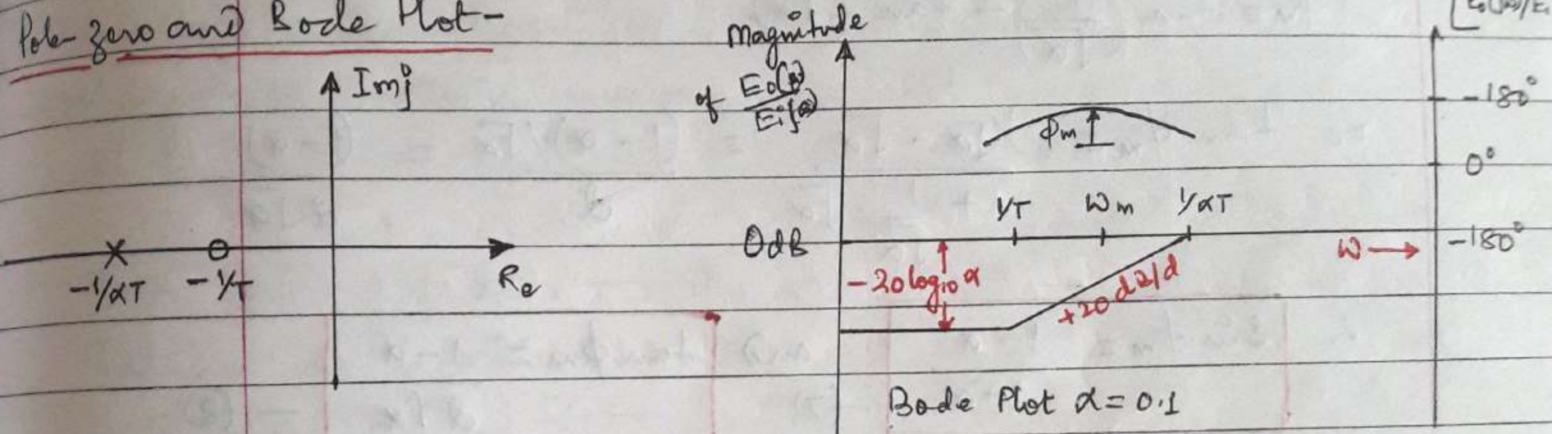
- The steady state performance of the T.F. is governed by low freq. characteristics and transient performance is governed by high freq. characteristics. For simultaneous satisfaction of transient & steady state performance, the Bode Plot must be reshaped so that the high freq. portion of the plot satisfies the phase margin requirement and low freq. portion satisfies the K_v requirement.

Pole-zero and Bode Plot-



Bode Plot $\alpha = 0.1$

Corner frequencies are $\omega = 1/T$ = lower corner freq. $= \omega_m\sqrt{\alpha}$

$\omega = 1/\alpha T$ = upper corner freq. $= \frac{\omega_m\sqrt{\alpha}}{\alpha} = \omega_m/\sqrt{\alpha}$

- from the pole-zero plot, it is clear that zero is nearer to the imaginary axis as compared to the pole. Zero being more dominant than pole, gives a +ve phase shift [Phase shift increases], hence is called phase lead N/w.

- Phase lead N/w allows to pass high fequencies and low fequencies are attenuated.
- Max^m phase lead occurs at mid-frequency ω_m between upper & lower corner frequencies, i.e. ω_m is the geometric mean of two corner frequencies

$$\therefore \quad \log_{10}\omega_m = \frac{1}{2}\left[\log_{10}\left(\frac{1}{\alpha T}\right) + \log_{10}\left(\frac{1}{T}\right)\right]$$

$$= \left[\log \frac{1}{\alpha T}\cdot\frac{1}{T}\right]^{1/2} = \log\left[\frac{1}{\alpha T^2}\right]^{1/2}$$

$$\boxed{\omega_m = \frac{1}{T\sqrt{\alpha}}} \quad \text{——} \quad (6)$$

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Phase angle $\angle E_o(j\omega)/E_i(j\omega)$ can be calculated as -

$$\angle \frac{E_o(j\omega)}{E_i(j\omega)} = \tan^{-1}(\omega T) - \tan^{-1}(\omega \alpha T)$$

At $\omega = \omega_m = 1/\sqrt{\alpha}\,T$ the phase angle is ϕ_m

$$\therefore \phi_m = \tan^{-1}\left(\frac{1}{T\sqrt{\alpha}}T\right) - \tan^{-1}\left(\frac{1}{\sqrt{\alpha}\,T}\cdot \alpha T\right)$$

$$\phi_m = \tan^{-1}\left(\frac{1}{\sqrt{\alpha}}\right) - \tan^{-1}(\sqrt{\alpha})$$

$$\text{or } \tan\phi_m = \frac{1/\sqrt{\alpha} - \sqrt{\alpha}}{1 + \frac{1}{\sqrt{\alpha}}\cdot\sqrt{\alpha}} = \frac{(1-\alpha)/\sqrt{\alpha}}{2} = \frac{(1-\alpha)}{2\sqrt{\alpha}}$$

$$\therefore \boxed{\sin\phi_m = \frac{1-\alpha}{1+\alpha}} \quad -(7) \quad \text{and} \quad \boxed{\tan\phi_m = \frac{1-\alpha}{2\sqrt{\alpha}}} \quad -(8)$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Design Procedure for Phase Lead Compensation -

Step 1 - The magnitude and phase Vs freq. curves are plotted for G(s) of the uncompensated system with gain constant K set according to steady state error requirement.

Step 2 - From the Bode Plot determine the phase and gain margins.

Let ϕ = Phase margin of uncompensated system.

ϕ_s = Specified phase margin

ϵ = Margin of safety.

then, $\phi_m = \phi_s - \phi + \epsilon$

If $\phi_m > 60^\circ$, two identical N/w each contributing max^m lead of $\phi_m/2$ are used

Step 3 - Use eq^n (7) to calculate the value of 'α'

Step 4 - Calculate ω_m at which uncompensated system will have a gain equal to $-10\log\alpha$ [the phase lead N/w causes an attenuation of $-\frac{1}{2}(20\log\alpha) = -10\log\alpha$ dB at freq. ω_m]

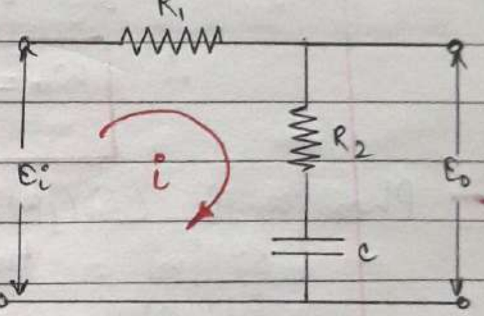
Step 5 – Once 'α' is determined, calculate the value of T from $\omega_m = 1/\sqrt{\alpha}\,T$

Step 6 – Transfer function of phase lead N/w is determined from the values of 'α' and 'T'.

Step 7 – Draw the Bode Plot of the compensated system and check that all performance specifications are met or not. If not, a new value of ϕ_m must be estimated.

2) Phase-Lag Compensation –

Applying KVL in both the meshes –



$$E_i = R_1 i + R_2 i + \frac{1}{C}\int i\,dt \quad —(1)$$

$$E_o = R_2 i + \frac{1}{C}\int i\,dt \quad —(2)$$

Taking L.T. of eq^n (1) & (2) –

$$E_i(s) = (R_1 + R_2)\,I(s) + \frac{1}{Cs}\,I(s) \quad —(3)$$

$$E_o(s) = R_2\,I(s) + \frac{1}{Cs}\,I(s) \quad —(4)$$

Now
$$\frac{E_o(s)}{E_i(s)} = \frac{(R_2 + 1/Cs)\,\cancel{I(s)}}{\left[(R_1+R_2) + \frac{1}{Cs}\right]\cancel{I(s)}} = \frac{1 + R_2 Cs}{(R_1+R_2)Cs + 1} \quad —(5)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + 1/sC}{R_1 + R_2 + \frac{1}{sC}} = \frac{1}{\left(\frac{R_1+R_2}{R_2}\right)}\left[\frac{s + (1/R_2 C)}{s + \frac{1}{\left(\frac{R_1+R_2}{R_2}\right)R_2 C}}\right] \quad —(6)$$

Let $T = R_2 C$ and $\beta = \left(\frac{R_1+R_2}{R_2}\right) > 1$

or
$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1+sT}{1+s\beta T}} \quad —(7)$$

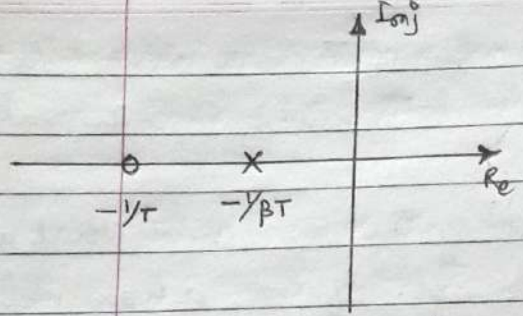
In sinusoidal form eq^n (7) can be expressed as –

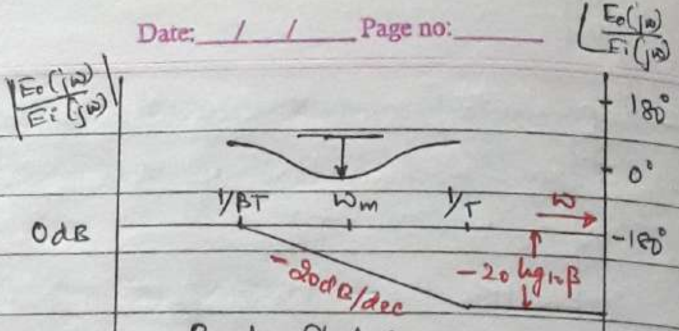
$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{1+j\omega T}{1+j\omega\beta T} \quad —(8)$$

– Two corner frequencies are given by –

$\omega = 1/T$ upper corner freq. =

$\omega = 1/\beta T$ lower corner freq. =





Bode Plot $\beta = 10$

- The max^m phase lag ϕ_m occurs at mid freq. ω_m between upper and lower corner frequencies.

$$\therefore \log_{10}\omega_m = \frac{1}{2}\left[\log_{10}(1/\beta T) + \log_{10}(1/T)\right]$$

$$\therefore \boxed{\omega_m = \frac{1}{T\sqrt{\beta}}} \quad \text{— (9)}$$

- Phase angle $\angle E_o(j\omega)/E_i(j\omega)$ can be calculated as-

$$\angle \frac{E_o(j\omega)}{E_i(j\omega)} = \tan^{-1}(\omega T) - \tan^{-1}(\omega\beta T)$$

At $\omega = \omega_m = 1/\sqrt{\beta}\,T$, the phase angle is ϕ_m

$$\therefore \tan\phi_m = \frac{1-\beta}{2\sqrt{\beta}} \quad \text{— (10)}$$

$$\text{and } \sin\phi_m = \frac{1-\beta}{1+\beta} \quad \text{— (11)}$$

- From pole-zero plot it can be seen that pole is nearer to origin hence it is dominant, thereby reduces the phase shift. Hence when phase lag N/w is introduced in cascade with forward path of a T.F., phase shift is reduced.

- The Bode plot of phase lag N/w reveals that it allows to pass the low frequency and high frequencies are attenuated.

Design Process for Phase Lag Network-

Step 1- Same as that of the phase lead N/w compensation.

Step 2 – From the Bode Plot, determine phase margin of uncompensated system.

Step 3 – If ϕ_s = specified phase margin

ϵ = margin of Safety

$\phi = \phi_s + \epsilon$

Step 4 – Determine the freq. corresponding to the reqd. phase margin which is the new gain crossover frequency (ω_m')

Step 5 – The magnitude curve is brought down to 0 dB at the new gain crossover freq. where the P.M. is satisfied, the phase lag N/w must provide the amount of attenuation equal to the value of magnitude curve at ω_m'.

$$|G(j\omega_m')| = -20\log\beta \qquad a<1$$

$$\text{or } \beta = 10^{-|G(j\omega_m')|/20} \qquad a<1$$

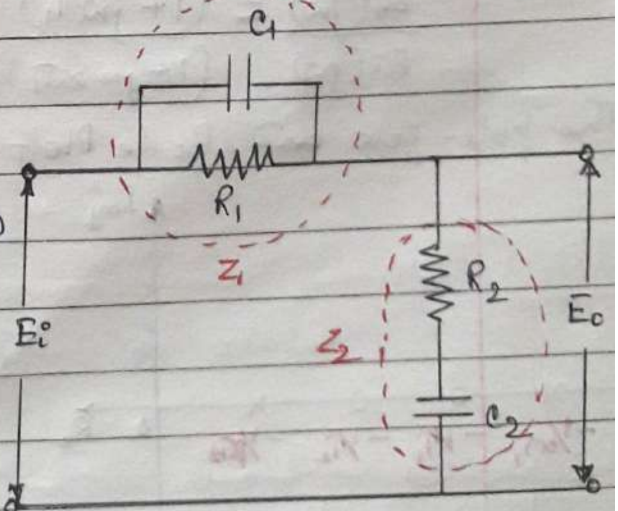
calculate 'β' from above expression.

Step 6 – Calculate T from → $1/\beta T = \omega_m'/10$

Usually lower corner freq ($1/\beta T$) is placed at a frequency about one decade below the new gain crossover freq.

Step 7 – Same as that of phase lead N/w

Phase Lead – Lag Compensation –



– In phase lead Compensation, the gain crossover freq. shifts to a higher value, thus bandwidth increase, speed of responce becomes fast but steady state error doesn't shows much improvement.

– In phase lag, GCF shifts to a lower value, BW decreases, speed of responce reduces but the steady state error improve

– The speed of response and steady state error can be simultaneously improved if both phase lead and lag compensation N/ws are used.

From the diagram, applying KVL to the meshes –

$$E_i = Z_1 i + Z_2 i = (Z_1 + Z_2)\, i \qquad \text{— (1)}$$

and

$$E_0 = Z_2 i \qquad \text{— (2)}$$

Taking Laplace Transform of above two eq^{n}s –

$$E_i(s) = [Z_1(s) + Z_2(s)]\, I(s) \qquad \text{— (3)}$$

$$E_0(s) = Z_2(s)\, I(s) \qquad \text{— (4)}$$

Now
$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{(1 + R_2 C_2 s)/C_2 s}{\left(\frac{1 + R_2 C_2 s}{C_2 s}\right) + \frac{R_1}{1 + R_1 C_1 s}}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_1 C_1 s)(1 + R_2 C_2 s)}{1 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + R_1 R_2 C_1 C_2 s^2} \qquad \text{— (5)}$$

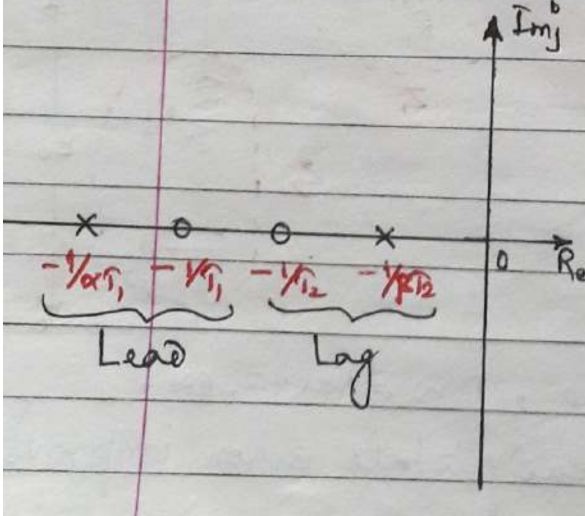
Now $T_1 = R_1 C_1$, $T_2 = R_2 C_2$, $\alpha < 1$, $\beta > 1$, $\alpha\beta = 1$ — (6)

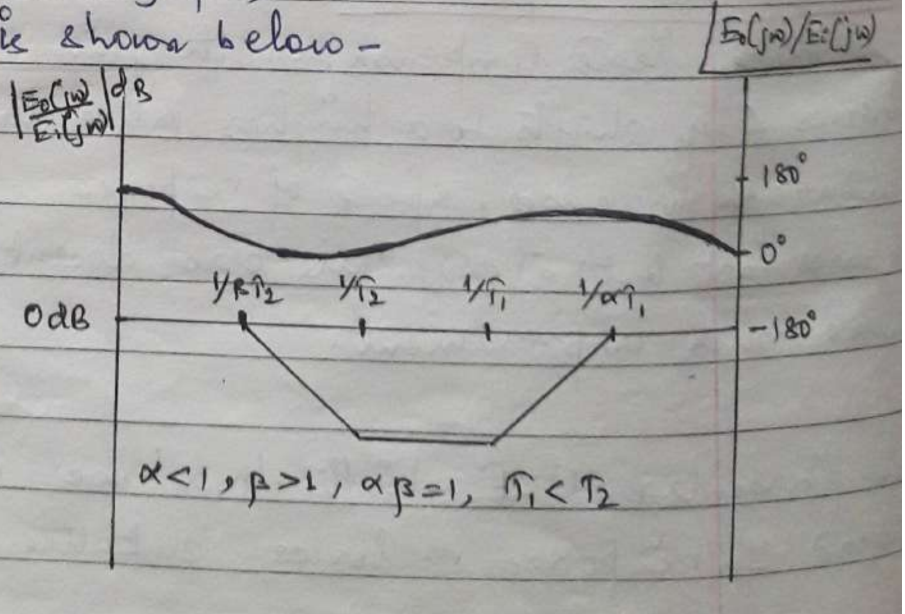
$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + sT_1)}{(1 + s\alpha T_1)} \times \frac{(1 + sT_2)}{(1 + s\beta T_2)} \qquad \text{— (7)}$$

In sinusoidal form, the transfer function can be written as –

$$\frac{E_0(j\omega)}{E_i(j\omega)} = \frac{(1 + j\omega T_1)}{(1 + j\omega\alpha T_1)} \times \frac{(1 + j\omega T_2)}{(1 + j\omega\beta T_2)}$$

The pole-zero and Bode Plot is shown below –





$\alpha < 1$, $\beta > 1$, $\alpha\beta = 1$, $T_1 < T_2$

Q1. Design a cascade compensation for a system whose T.F. is given by-

$$G(s) = \frac{K}{s(1+0.1s)(1+0.001s)}$$

fulfilling the foll^{n}g specifications — a) P.M. $\geq 45°$ b) $K_v = 1000 \text{ sec}^{-1}$

Sol^n Step 1- $K_v = \lim_{s \to 0} s\,G(s) = \lim_{s \to 0} \cancel{s} \cdot \frac{K}{\cancel{s}(1+0.1s)(1+0.001s)}$

$K_v = K = 1000$

$$\therefore \quad G(s) = \frac{1000}{s(1+0.1s)(1+0.001s)}$$

Step 2 Initial slope of Bode Plot is -20 dB/dec [Type '1' system] which intersects the 0 dB axis at $\omega = K = 1000$.

Corner frequencies are $\omega_1 = 1/0.1 = 10$ rad/sec and $\omega_2 = 1/0.001 = 1000$

$\phi = -90° - \tan^{-1}(0.1\omega) - \tan^{-1}(0.001\omega)$

ω	1	5	10	50	100	150	200	500
ϕ	-95.7	-116.5	-135.63°	-171.46°	-179.9°	-184°	-188.43°	-205.41°

Step 3- From the Bode plot :- Available phase margin $\phi = 0°$

specified phase margin $\phi_s = 45°$

Margin of safety $= 5° = \epsilon_s$

$\therefore \quad \phi_m = \phi_s - \phi + \epsilon_s = 45° - 0 + 5° = 50°$

Step 4- Calculation of 'α' $\rightarrow$ $\sin\phi_m = \frac{\alpha - 1}{\alpha + 1}$

$\sin 50° = \frac{\alpha-1}{\alpha+1} \Rightarrow 0.766 = \frac{\alpha-1}{\alpha+1} \Rightarrow 0.766(\alpha+1) = \alpha - 1$

$\Rightarrow 0.766\alpha + 0.766 = \alpha - 1 \Rightarrow 1 + 0.766 = \alpha - 0.766\alpha = 0.234\alpha$

or $\alpha = \frac{1.766}{0.234} = 7.51$

Step 5:- Calculation of ω_m —

Zero freq. attenuation $= -10\log\alpha = -10\log(7.51) = -8.75\,dB$

At the gain of $-8.75\,dB$ draw a line on magnitude curve, which gives ω_m (new GCF)

— From the Bode Plot, it is noted for the uncompensated system at a gain of $-8.75\,dB$, the freq is 150 rad/sec = New GCF

$$\therefore \omega_m = 150\ \text{rad/sec}$$

Step 6:- Calculation of T —

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \qquad \text{Now } T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{150\sqrt{7.51}} = 0.00243$$

Step 7:- Transfer function of Compensator —

$$G_c(s) = \frac{E_o(j\omega)}{E_i(j\omega)} = \frac{[1+j\omega T]}{[1+j\omega\alpha T]}$$

Amplifier gain is $1/\alpha = 1/7.51$

Lower corner freq $= 1/T = 1/0.00243 = 411.5$

Upper corner freq $= 1/\alpha T = \frac{\sqrt{\alpha}\,\omega_m}{\alpha} = \frac{\omega_m}{\sqrt{\alpha}} = \frac{150}{\sqrt{7.51}} = 54.74$

$$\text{Now } G_c(s) = \frac{1+j\omega\cdot 1/411.5}{1+j\omega\times 7.51\times\frac{1}{411.5}} = \frac{1+j\,0.00243}{1+j\,0.018}$$

Amplification necessary to cancel the lead n/w attenuation of 7.51

$$\therefore G_c(s) = \frac{1+j\,0.00243}{1+j\,0.018}$$

Step 8: Overall transfer function —

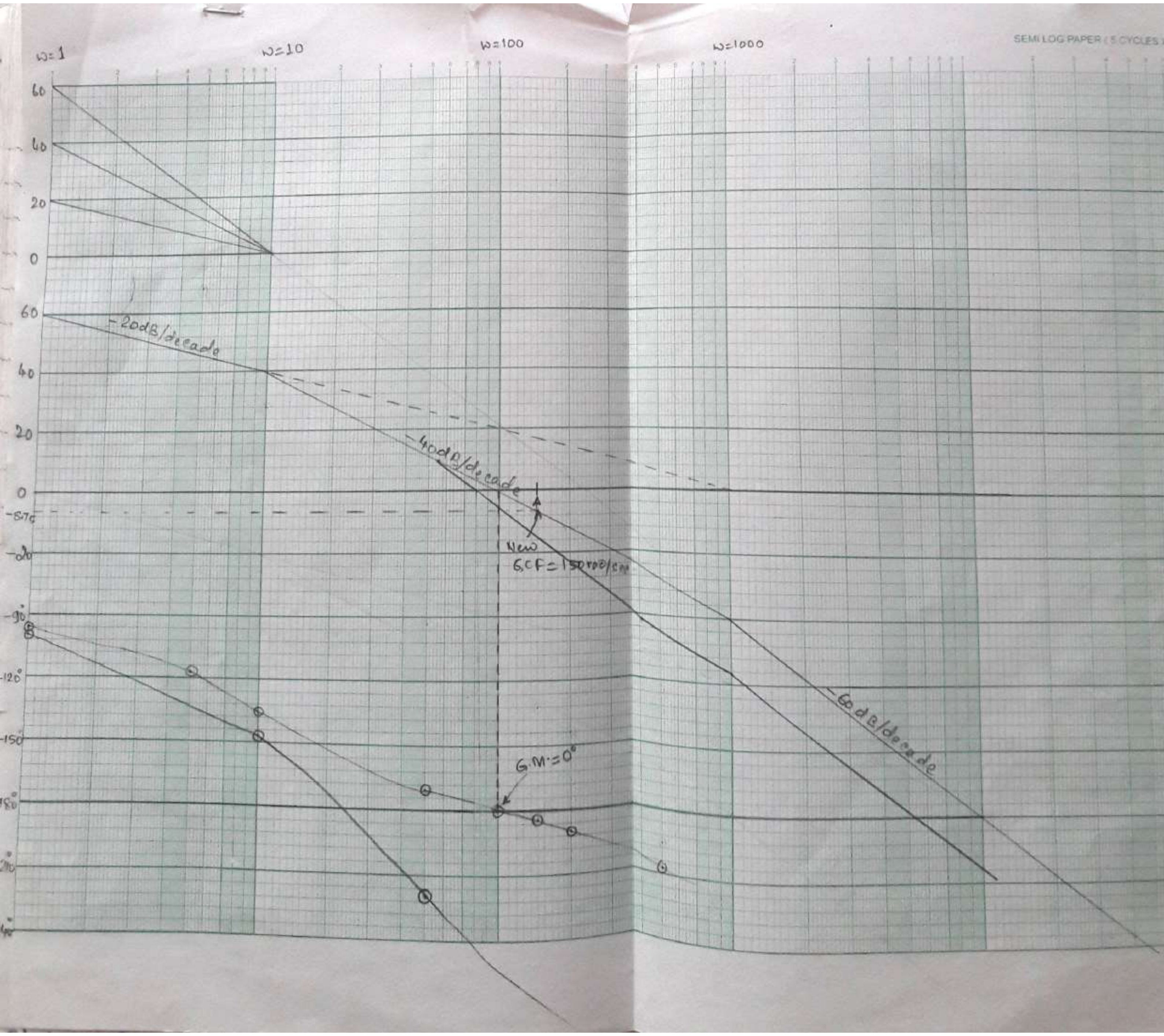
$$G'(s) = G_c(s)\cdot G(s) = \frac{(1+j\,0.00243s)}{(1+j\,0.018s)} \times \frac{1000}{s(1+0.1s)(1+0.001s)}$$

Step 9:- Corner frequencies — $\omega_1 = 10$ rad/sec, $\omega_2 = 1/0.018 = 55.5$ rad/sec

$\omega_3 = 1/0.00243 = 411.5$ rad/sec, $\omega_4 = 1/0.001 = 1000$ rad/sec

Phase angle $\phi = -90^\circ + \tan^{-1}(0.00243\omega) - \tan^{-1}(0.1\omega) - \tan^{-1}(0.018\omega) -$

$\tan^{-1}(0.001\omega)$



SEMI LOG PAPER (5 CYCLES)
ω=1
ω=10
ω=100
ω=1000
60
40
20
0
-20dB/decade
-40dB/decade
-60dB/decade
New GCF=150rad/sec
G.M=0°
-90°
-120°
-150°
-180°

ω	1	10	50	100	150	200
φ	-96.9°	-147.2	-220.4°	-254.6°	-274.4	-288.8°

Q2. The O.L.T.F. of a UFB control system is $G(s) = K/s(1+0.2s)$. Design a suitable compensator such that the system will have $K_v = 10$ & P.M $= 50°$

Soln- Step 1: $K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s. \frac{K}{s(1+0.2s)} = K$

or $K = K_v = 10$

$\therefore$ OLTF becomes $G(s) = \frac{10}{s(1+0.2s)}$

Step 2 - Initial slope of Bode Plot is -20dB/decade. Corner frequencies $\omega_1 = 1/0.2 = 5$ rad/sec, intersection with 0dB axis at $\omega = k =$

Phase angle, $\phi = -90° - \tan^{-1}(0.2\omega)$

ω	1	5	10	50	100	200	500	1000	2000
φ	-101.3	-135°	-153.3	-174.3	-177.1	-178.6	-179.4	-179.7	-179.8

Step 3:- From the Bode Plot :- Available P.M $= 180° - 144° = 36°$

and Gain cross over freq $= 6.7$ rad/sec

specified P.M $= 50°$, Margin of safety $= 6°$

$\therefore \phi_m = \phi_s - \phi + \epsilon = 50° - 36° + 6° = 20°$

Step 4:- Calculation of $\alpha \rightarrow$

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} = \frac{1 - \sin 20°}{1 + \sin 20°} = 0.49$$

Step 5:- Calculation of ω_m

Zero freq. attenuation $= -10 \log \alpha = -10 \log(0.49) = -3.1$ dB.
From the Bode plot for uncompensated system at a gain of -3.1dB the frequency is 8.2 rad/sec. Hence the new gain crossover frequency for the compensated system is selected at $\omega = \omega_m = 8.2$ rad/sec

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Step 6:- Corner frequencies for Compensated system -

lower corner frequency $= 1/T = \sqrt{\alpha}\,\omega_m = \sqrt{0.49} \times 8.2 = 5.74$ rad/s

upper corner frequency $= 1/\alpha T = \frac{\omega_m}{\sqrt{\alpha}} = \frac{8.2}{\sqrt{0.49}} = 11.7$ rad/sec.

Step 7:- The T.F. for the phase lead compensation n/w with amplifier gain

$1/\alpha = 1/0.49 = 2.04$ is

$$G_c(j\omega) = \frac{1+j\omega T}{1+j\omega\alpha T} = \frac{1+j\omega\, 1/5.74}{1+j\omega \times 0.49 \times 1/5.74} = \frac{1+j\omega 0.174}{1+j0.085\omega}$$

Step 8:- Over all Transfer function -

$$G'(s) = G_c(s)\,G(s) = \left(\frac{1+j0.174\omega}{1+j0.085\omega}\right) \times \frac{10}{s(1+0.2s)}$$

$$G'(s) = \frac{(1+0.174s)}{(1+0.085s)} \times \frac{10}{s(1+0.2s)}$$

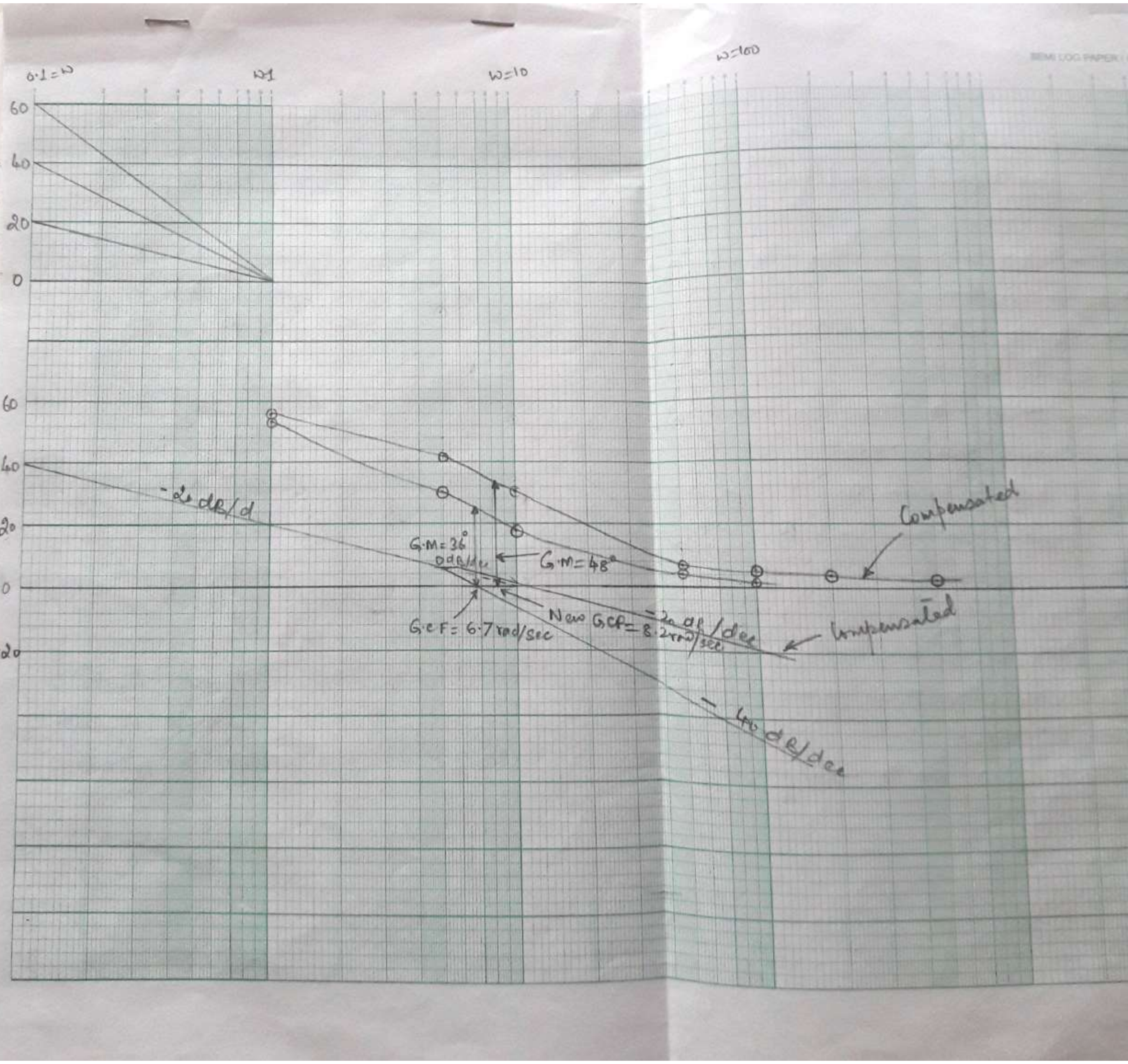
Initial slope = −20 dB/dec. upto 5 rad/sec

change of slope = −20 + 20 = 0 dB/dec due to 5.74 rad/sec.

change of slope = 0 − 20 = −20 dB/dec due to 11.7 rad/sec.

Phase angle $\phi = -90^\circ - \tan^{-1}(0.085\omega) - \tan^{-1}(0.2\omega) + \tan^{-1}(0.174\omega)$

ω	1	5	10	50	100	200	500
ϕ	−96.3°	−117°	−133.6°	−167.6°	−173.7°	−176.8°	−178.7°



0.1=ω
ω1
ω=10
ω=100
SEMI LOG PAPER
60
40
20
0
-20
-20 dB/d
G.M=36°
0dB/dec
G.M=48°
G.C.F=6.7 rad/sec
New G.C.F=8.2 rad/sec
-20 dB/dec
-40 dB/dec
Compensated
Uncompensated

- STATE SPACE ANALYSIS OF CONTROL SYSTEMS -

- Transfer function approach of analysing the system has got some disadvantages like T.F. is defined only under zero initial conditions and is applicable to linear time invariant systems.

- State space approach of analysis can be applied to linear, non-linear, time invariant or time variant and MIMO systems.

- State space analysis involves the description of the system in terms of 1st order differential eq^ns by selecting suitable state variables.

- Advantages of state space approach -
  - Can be applied to linear, non-linear, time variant or invariant systems
  - It is easier to apply where the Laplace transform can't be applied.
  - n^{th} order differential eq^ns can be expressed as 'n' eq^ns of 1st order whose solutions are easier.
  - It is a time domain approach.
  - This method is suitable for digital computer computations because it is a time domain approach.
  - The system can be designed for optimal conditions w.r.t. given performance indices.

DEFINITIONS -

1) State - The state of a system at any time 't_0' is the minimum set of numbers $x_1, x_2, \ldots x_n$ which along with the input to the system for time $t \geq t_0$ is sufficient to determine the behaviour of the system for all $t \geq t_0$

2) State Variables - The smallest set of variables which determine the state of a dynamic system are called state variables.

3) space vector – If n state variables are necessary to determine the behaviour of a given system, the variables can be considered as 'n' components of a vector called state vector.

4) State Space – The n-dimensional state variables are elements of n-dimensional space called state space. or
The n-dimensional space whose coordinate axes consists of the x_1, x_2, x_n axes is called state space.

State space Representation of Systems –

1) n^{th} order differential eq^n – The eq^n is given by –

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = u(t) \quad \text{—— (1)}$$

- The input $u(t)$ and the initial cond's $y(0), \frac{dy(0)}{dt} \ldots \frac{d^{n-1}y(0)}{dt^{n-1}}$ at $t=0$ are given.
- Dynamic behaviour of the system can be determined from the knowledge of $u(t), y(t), \cdots y^{n-1}(t)$.
- The terms $y(t), \dot{y}(t), \cdots y^{n-1}(t)$ can be considered as a set of n-state variables
- Let $y = x_1;\ \dot{y} = x_2, \cdots\ y^{n-1} = x_n$ —— (1a)
- Set of eq^n (1a) can be written as –

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n \quad \text{—— (2)}$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \cdots - a_1 x_{n-1} + u$$

- A general representation of set of eq^n (2) can be expressed in the form of state eq^{n}s as – $\boxed{\dot{x} = Ax + Bu}$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$; $A = \begin{bmatrix} 0 & 1 & \cdots 0 & 0 \\ 0 & 0 & \cdots 1 & 0 \\ 0 & 0 & \cdots 0 & 1 \\ \vdots & & & \\ -a_n & -a_{n-1} & \cdots 0 & a_1 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ —— (3)

where x = state vector $(n \times 1)$; u = Control (scalar)
A = matrix $(n \times n)$; B = matrix $(n \times 1)$

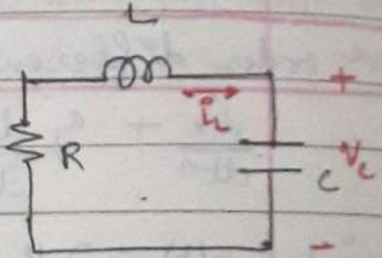
— The output eq^n is given by — $y = Cx$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{—(4)}$$

where $C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$; y = Output (scalar), C = matrix $(1 \times n)$

2) State space representation for Electrical Network —

— The variable i_L and V_c are called state variables of the network.



Apply KVL — $R i_L + L \dfrac{di_L}{dt} + V_c = 0$ — (1)

Also $i_c = i_L = C \dfrac{dv_c}{dt}$ — (2)

from eq^n(1) → $\dfrac{di_L}{dt} = -\dfrac{R}{L} i_L - \dfrac{1}{L} V_c$ —(3)

and $\dfrac{dv_c}{dt} = \dfrac{1}{C} i_L$ —(4)

Let $x_1 = i_L$ and $x_2 = V_c$

$\therefore \quad \dot{x}_1 = -\dfrac{R}{L} x_1 - \dfrac{1}{L} x_2$ —(5)

and $\dot{x}_2 = \dfrac{1}{C} x_1$ —(6)

In matrix form $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

3) State space representation for Transfer function —

$$G(s) = \frac{y(s)}{u(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}$$

This transfer function has no zeros.

$$(s^3 + a_3 s^2 + a_2 s + a_1)\, y(s) = K\, u(s)$$

or $$s^3 y(s) + a_3 s^2 y(s) + a_2 s\, y(s) + a_1 y(s) = K\, u(s)$$

Taking inverse Laplace transform -

$$\dddot{y}(t) + a_3 \ddot{y}(t) + a_2 \dot{y}(t) + a_1 y(t) = K\, u(t)$$

or $$\dddot{y}(t) = K u(t) - a_3 \ddot{y}(t) - a_2 \dot{y}(t) - a_1 y(t)$$

select the state variables as, first state variable as output

$$y(t) = x_1$$
$$\dot{y}(t) = \dot{x}_1 = x_2$$
$$\ddot{y}(t) = \dot{x}_2 = x_3$$
$$\dddot{y}(t) = \dot{x}_3$$

$\therefore$ $$\dddot{y}(t) = -a_3 x_3 - a_2 x_2 - a_1 x_1 + K u(t)$$

Rewriting the eq^ns -

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_3 x_3 - a_2 x_2 - a_1 x_1 + K u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u(t)$$

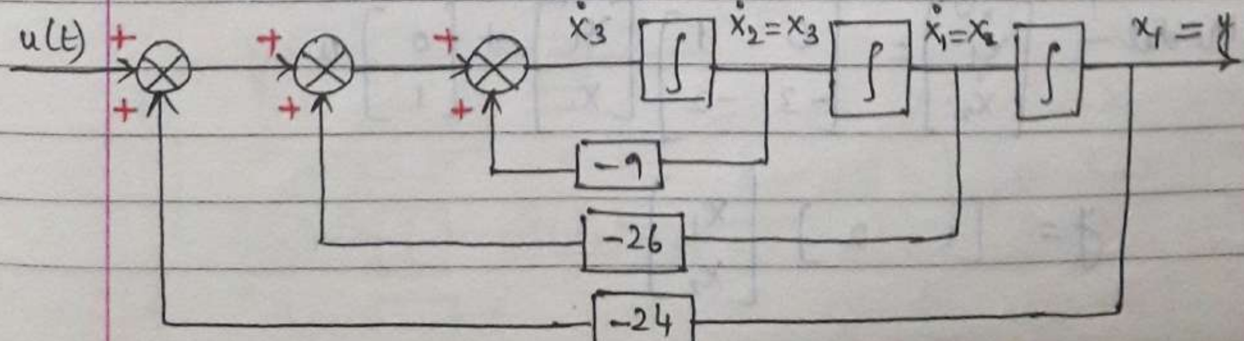
$$y(t) = [1\ 0\ 0]\, x_1(t)$$

Transfer function Decomposition -

1) Direct Decomposition -

Direct decomposition of a T.F is performed for the system, when the T.F is given in the following form -

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 9s^2 + 26s + 24}$$



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + u$$

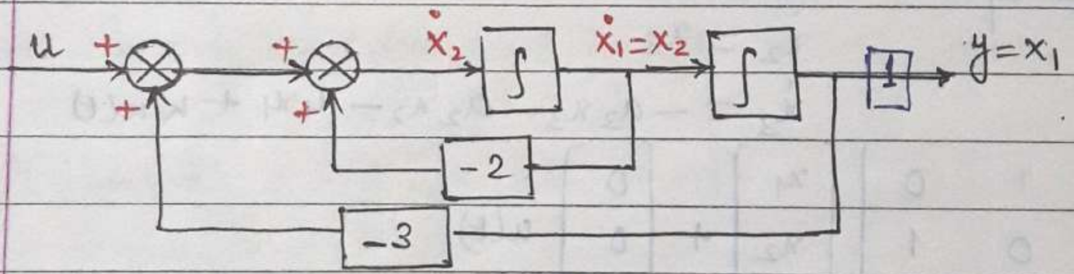
and o/p eqⁿ is — $y = x_1$

In matrix from — $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$

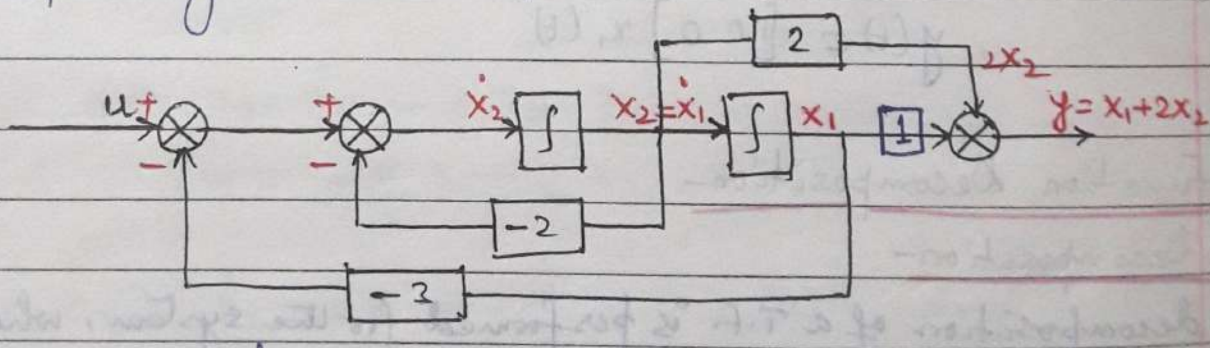
and $[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

— If the transfer funcⁿ is given by $\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+2s+3}$

split the T.F. into two parts $\frac{Y(s)}{U(s)} = \frac{1}{s^2+2s+3} + \frac{2s}{s^2+2s+3}$



Now incorporating the term $2s/s^2+2s+3$



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -3x_1 - 2x_2 + u$$

o/p eqⁿ $y = x_1 + 2x_2$

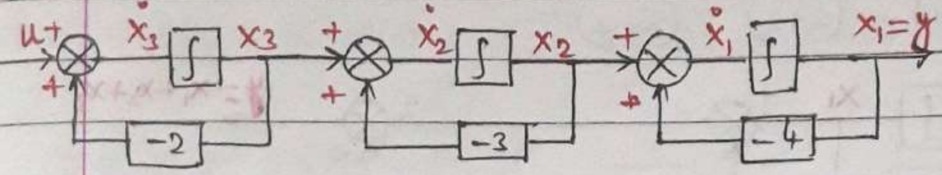
State Model — $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2) Cascade Decomposition-

The transfer function is given in the following form-

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+2)} \cdot \frac{1}{(s+3)} \cdot \frac{1}{(s+4)}$$



Output of each integrator is assigned a state variable and state eq^n's are

$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + u$$

O/P eq^n is $y = x_1$

State Model is -

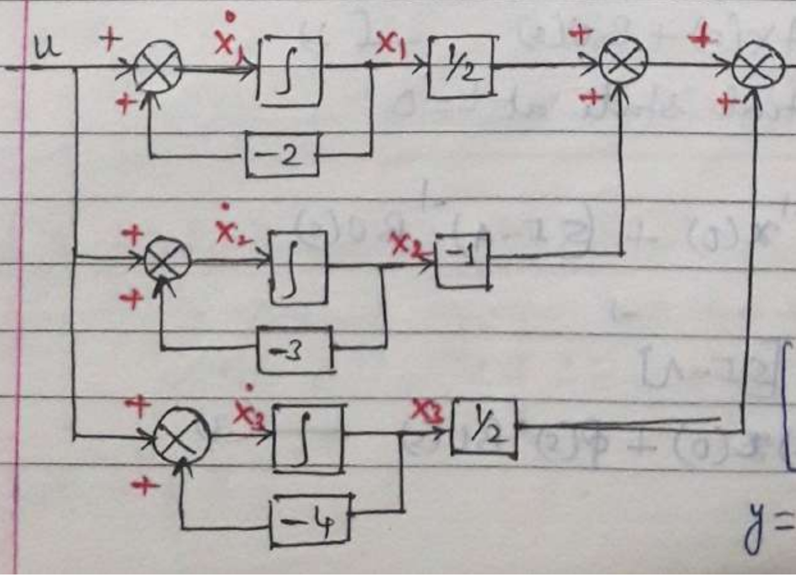
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3) Parallel Decomposition-

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+2)(s+3)(s+4)}$$ split into partial fractions

$$\frac{Y(s)}{U(s)} = \frac{1/2}{(s+2)} + \frac{-1}{(s+3)} + \frac{1/2}{(s+4)}$$



$$y = \frac{1}{2}x_1 - x_2 + \frac{1}{2}x_3$$

$$\dot{x}_1 = -2x_1 + u$$
$$\dot{x}_2 = -3x_2 + u$$
$$\dot{x}_3 = -4x_3 + u$$

$$y = \tfrac{1}{2}x_1 - x_2 + \tfrac{1}{2}x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

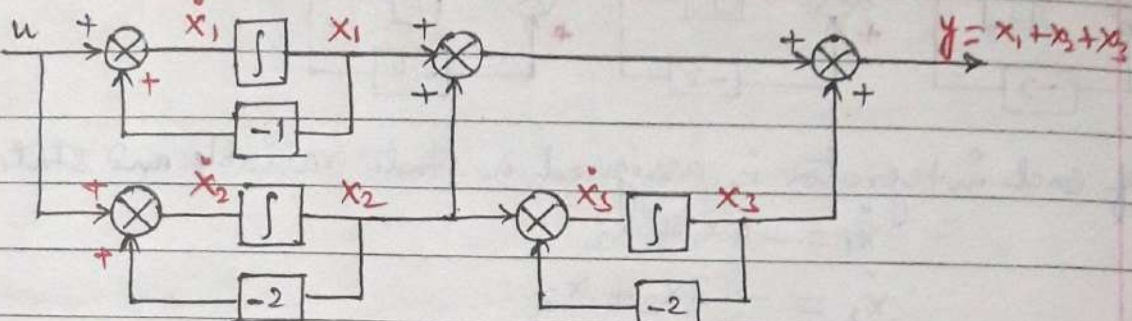
$$y = \begin{bmatrix} 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4) Parallel decomposition of a T.F. with repeated roots of its charact. eqⁿ –

$$\frac{Y(s)}{U(s)} = \frac{2s^2+8s+7}{(s+2)^2(s+1)}$$

Splitting the RHS of the T.F. into partial fractions –

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+1)} + \frac{1}{(s+2)} + \frac{1}{(s+2)^2}$$



$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = -2x_2 + u \qquad \text{and O/P eq}^n \text{ is } y = x_1 + x_2 + x_3$$
$$\dot{x}_3 = x_2 + 2x_3$$

State Model is –

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 0\end{bmatrix} u$$

$$y = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Solution of State Equation –

The State eqⁿ is given by $\frac{dx(t)}{dt} = Ax(t) + Bu(t)$ —— (1)

Taking L.T. on both sides –

$$sX(s) - x(0) = AX(s) + BU(s) \quad \text{—— (2)}$$

where $x(0)$ = initial state at $t=0$

— Rearranging eqⁿ(2) –

$$X(s) = [sI - A]^{-1}x(0) + (sI - A)^{-1}BU(s)$$

Put $\phi(s) = [sI - A]^{-1}$

$$\therefore \; X(s) = \phi(s)x(0) + \phi(s)BU(s) \quad \text{—— (3)}$$

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Taking Laplace inverse transform on both sides of eq^n (3) -

$$\mathcal{L}^{-1} X(s) = \mathcal{L}^{-1} \phi(s)\, x(0) + \mathcal{L}^{-1} \phi(s)\, BU(s)$$

$$\boxed{x(t) = \underbrace{\phi(t)\, x(0)}_{ZIR} + \underbrace{\mathcal{L}^{-1} \phi(s)\, BU(s)}_{ZSR}}$$

where $\phi(t) = \mathcal{L}^{-1} \phi(s) = \mathcal{L}^{-1} [sI - A]^{-1}$ = state transition Matrix

OR

$\dot{x}(t) = A x(t) + Bu(t)$ —— (1)

$u(t) = 0$ for unforced response

Then $\dot{x}(t) = A x(t)$ —— (2)

Consider the analogous scalar equation

$\dot{x}(t) = a x(t)$ —— (3)

Taking Laplace transform of eq^n (3) -

$sX(s) - x(0) = aX(s)$

$(s-a) X(s) = x(0)$

or $X(s) = (s-a)^{-1} x(0)$ —— (4)

Taking inverse Laplace transform of eq^n (4) -

$x(t) = e^{at} x(0)$ —— (5)

If eq^n (5) is the solution of (4) then sol^n of eq^n (2) is

$$\boxed{x(t) = e^{At} x(0)}$$

$e^{At} = \phi(t)$ = State transition matrix

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{i=0}^{n} \frac{A^i t^i}{i!} \quad \text{—— (6)}$$

$$\boxed{\phi(t) = \mathcal{L}^{-1} \phi(s) = \mathcal{L}^{-1} [sI - A]^{-1}} \quad \text{—— (7)}$$

Properties of state Transition Matrix -

for time invariant system $\dot{x} = Ax$ and

$$\phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

i) $\phi(0) = e^{A(0)} = I$

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ii) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

iii) $\phi^{-1}(t) = \phi(-t)$

iv) $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1} e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

v) $[\phi(t)]^n = \phi(nt)$

vi) $\phi(t_2-t_1)\phi(t_1-t_0) = \phi(t_2-t_0) = \phi(t_1-t_0)\phi(t_2-t_1)$

vii) $\dot{\phi}(t) = A\phi(t)$

Transfer Matrix-

The matrix relating Laplace Transform of output to laplace transform of input of state space representation of a control system is known as transfer matrix.

The state eqn's are given by-

$$\dot{x} = Ax + Bu \quad \text{——} (1)$$

$$y = Cx + Du \quad \text{——} (2)$$

Taking Laplace transform on both sides of eqn (1) and (2) -

$$sX(s) - x(0) = AX(s) + BU(s) \quad \text{——} (3)$$

$$Y(s) = CX(s) + DU(s) \quad \text{——} (4)$$

As per definition of transfer function $x(0) = 0$

$$sX(s) = AX(s) + BU(s) \quad \text{——} (5)$$

$$X(s) = [sI-A]^{-1} BU(s) \quad \text{——} (6)$$

Substituting (6) in (4)-

$$Y(s) = C[sI-A]^{-1}BU(s) + DU(s)$$

$$Y(s) = [C[sI-A]^{-1}B + D]\,U(s) \quad \text{——} (7)$$

The transfer matrix is given by-

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = C[sI-A]^{-1}B + D} \quad \text{——} (8)$$

If D is a null matrix then transfer matrix becomes

$$\boxed{G(s) = C[sI-A]^{-1}B} \quad \text{——} (9)$$

$$G(s) = C\frac{adj(sI-A)}{|sI-A|}B + D \text{ or } \frac{C[adj(sI-A)B + |sI-A|D]}{|sI-A|} \quad \therefore |sI-A| = CE$$

→ Roots of C.E. are often referred to as eigen values of Matrix A.
→ Any non-zero vector p_i that satisfies the matrix eq^n $(\lambda_i I - A)p_i = 0$ [λ_i = Eigen value]

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Controllability –

– A system is said to be controllable if any initial state $x(t_0)$ or x_0 can be transferred to any final state $x(t_f)$ in a finite time interval $(t_f - t_0)$, $t \geq 0$ by some control u.

– The concept was introduced by Kalman, and for a linear time invariant continuous system described by the state eq^n:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is said to be completely controllable if and only if the rank of the controllability matrix is defined as

$$Q = [B : AB : A^2B : \dots : A^{n-1}B]$$

is equal to rank 'n'.

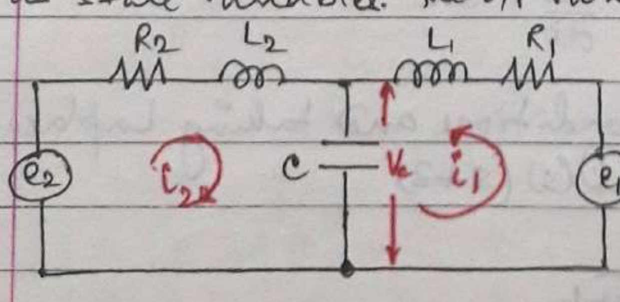
Observability :–

A system is said to be observable if every state x_0 can be exactly determined from the measurement of the output 'y' over a finite interval of time $0 \leq t \leq t_f$

– The system described by the above mentioned state eq^{n}s is said to be completely observable if the matrix of observability has 'n' rank where matrix is :–

$$V = [C^T : A^T C^T : (A^T)^2 C^T : \dots : (A^T)^{n-1} C^T]$$

Q1. For the shown electrical N/w, determine the state model. Consider i_1, i_2 and v_c as state variables. The o/p variables are x_1 and x_2.



Solⁿ - Differential eqⁿs are written as-

$$R_1 i_1 + L_1 \frac{di_1}{dt} + v_c = e_1$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} + v_c = e_2$$

and $$C \frac{dv_c}{dt} = i_1 + i_2$$

Rearranging the above eqⁿs:

$$\frac{di_1}{dt} = -\frac{R_1}{L_1} i_1 - \frac{1}{L_1} v_c + \frac{1}{L_1} e_1$$

$$\frac{di_2}{dt} = -\frac{R_2}{L_2} i_2 - \frac{1}{L_2} v_c + \frac{1}{L_2} e_2$$

$$\frac{dv_c}{dt} = \frac{1}{C} i_1 + \frac{1}{C} i_2$$

output eqⁿ. is :- $y_1 = i_1$ and $y_2 = i_2$

and Put $x_1 = i_1$, $x_2 = i_2$ and $x_3 = v_c$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & -1/L_2 \\ 1/C & 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/L_1 & 0 \\ 0 & 1/L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ 0 \end{bmatrix}$$

and $$[y] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q2. Obtain the state eqⁿs for the differential eqⁿ -

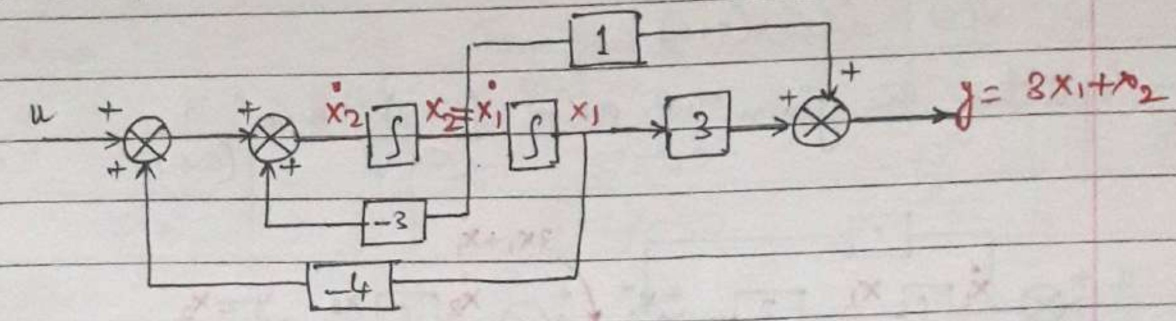
$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = \frac{du}{dt} + 3u$$

Solⁿ Assuming zero initial conditions and taking Laplace transform

$$Y(s)\left[s^2 + 3s + 4\right] = U(s)(s+3)$$

or $$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+4}$$

$$\frac{Y(s)}{U(s)} = \frac{3}{s^2+3s+4} + \frac{s}{s^2+3s+4}$$



Output of each integrator is assigned a state variable, so state eq^n s are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -3x_2 - 4x_1 + u$$

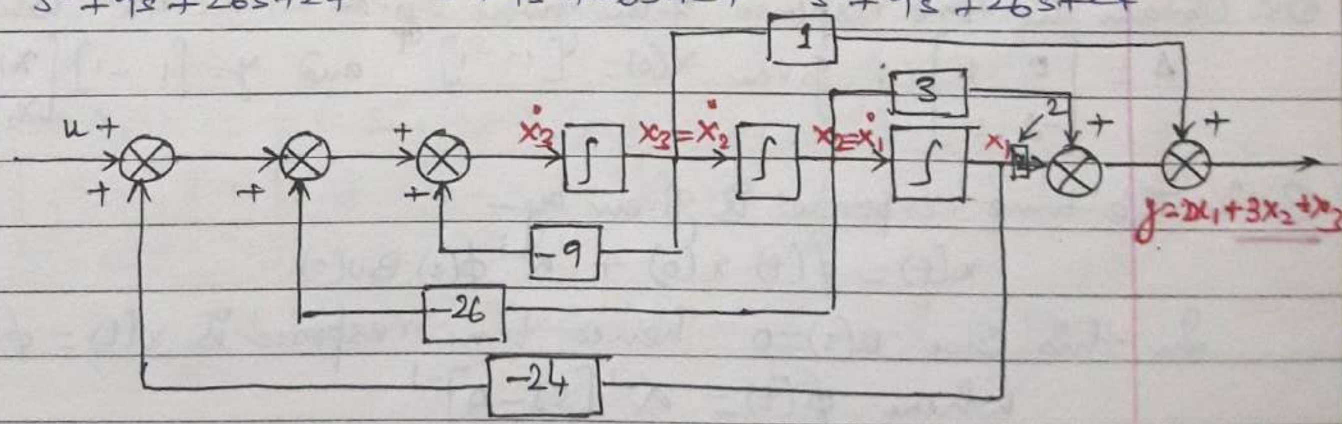
O/P eq^n is $y = 3x_1 + x_2$

State model is given by -

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{and} \quad [y] = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q3. The transfer function is given by $\frac{Y(s)}{U(s)} = \frac{s^2+3s+2}{s^3+9s^2+26s+24}$. Determine the state model. Use direct decomposition

Sol^n - $\frac{Y(s)}{U(s)} = \frac{2}{s^3+9s^2+26s+24} + \frac{3s}{s^3+9s^2+26s+24} + \frac{s^2}{s^3+9s^2+26s+24}$



from the diagram $\dot{x}_1 = x_2$

$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + u$$

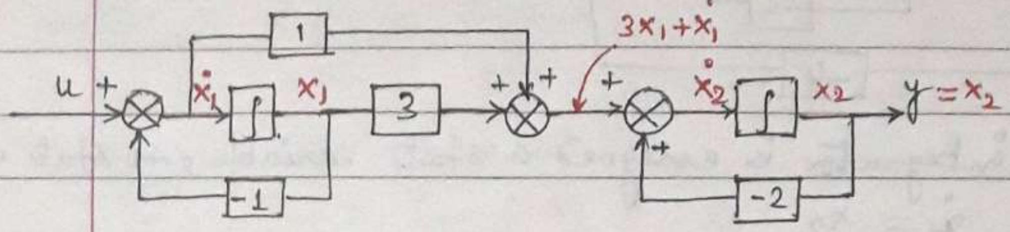
and $y = 2x_1 + 3x_2 + x_3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \;\; \& \;\; y = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q4. Using cascade method decompose the transfer func^n and obtain state model

$$\frac{Y(s)}{U(s)} = \frac{(s+3)}{(s+1)(s+2)}$$

Sol^n - splitting the given T.F. as - $\frac{(s+3)}{(s+1)}$ and $\frac{1}{(s+2)}$



$$\dot{x}_1 = -x_1 + u$$

$$\dot{x}_2 = 3x_1 + \dot{x}_1 - 2x_2 = 3x_1 - x_1 + u - 2x_2$$

or $\dot{x}_2 = 3x_1 - 2x_2 + u$

Output eq^n is - $y = x_2$

The state model is - $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$

and $[y] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Q5. Obtain the time response of the given system - $\dot{x} = Ax$ where $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$; given $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Sol^n- The time response is given by-

$$x(t) = \phi(t)\, x(0) + \mathcal{L}^{-1} \phi(s)\, BU(s)$$

In this case $U(s) = 0$ hence time response is $x(t) = \phi(t)\, x(0)$

where $\phi(t) = \mathcal{L}^{-1}[sI - A]^{-1}$

$$(sI - A) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}$$

Now $[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|} = \frac{\text{Transpose of Cofactor matrix}}{|sI - A|}$

$$|sI - A| = s^2 + 2$$

cofactors $a_{11} = s,\ a_{12} = -2,\ a_{21} = 1,\ a_{22} = s$

Cofactor Matrix $= \begin{bmatrix} s & -2 \\ 1 & s \end{bmatrix}$; Transpose of cofactor Matrix $= \begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}$

Now $(sI-A)^{-1} = \dfrac{1}{s^2+2}\begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} s/s^2+2 & 1/s^2+2 \\ -2/s^2+2 & s/s^2+2 \end{bmatrix} = \phi(s)$

Now $\phi(t) = \mathcal{L}^{-1}\phi(s) = L^{-1}\begin{bmatrix} \dfrac{s}{s^2+2} & \dfrac{1}{s^2+2} \\ \dfrac{-2}{s^2+2} & \dfrac{s}{s^2+2} \end{bmatrix} = L^{-1}\begin{bmatrix} \dfrac{s}{s^2+(\sqrt{2})^2} & \dfrac{1}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{s^2+(\sqrt{2})^2} \\ -\dfrac{2}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{s^2+(\sqrt{2})^2} & \dfrac{s}{s^2+(\sqrt{2})^2} \end{bmatrix}$

$$\phi(t) = \begin{bmatrix} \cos\sqrt{2}t & \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}$$

Now $x(t) = \phi(t)\cdot x(0) = \begin{bmatrix} \cos\sqrt{2}t & \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore\ x_1(t) = \cos\sqrt{2}t + \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t$

$x_2(t) = -\sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t$

$\because\ y = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 - x_2 = \left[\cos\sqrt{2}t + \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right] - \left[-\sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t\right]$

$y = \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t = \dfrac{3}{\sqrt{2}}\sin\sqrt{2}t$ Ans.

Q6. A linear time invariant system is characterized by the homogeneous state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Compute the solution of homogeneous eq^n, assume the initial state vector:

$$X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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Sol^n:- $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; Now $[SI-A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$

$$\text{Adj}[SI-A] = \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}^T = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$|SI-A| = (s-1)^2$$

$$[SI-A]^{-1} = \frac{\text{Adj}[SI-A]}{|SI-A|} = \frac{1}{(s-1)^2}\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

Now $L^{-1}[SI-A]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} = \phi(t)$

Now solution of homogeneous eq^n is-

$x(t) = L^{-1}\phi(s)\,x(0) = \phi(t)\,x(0) =$ Zero input response

$$x(t) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

Q7. Given $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

Find the unit step response when $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Sol^n- $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$(SI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{Adj}(SI-A) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|SI-A| = (s+3)s + 2 = s^2 + 3s + 2 = (s+1)(s+2)$$

$$(s) = [sI-A]^{-1} = \frac{Adj(sI-A)}{|sI-A|} = \begin{bmatrix} \frac{(s+3)}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix}$$

$$\phi(s) = \begin{bmatrix} \frac{2}{(s+1)} - \frac{1}{(s+2)} & \frac{1}{s+1} - \frac{1}{s+2} \\ -\frac{2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\phi(t) = L^{-1}[\phi(s)] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Now $ZIR = \phi(t) \cdot x(0)$

$$ZIR = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} + e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} - e^{-t} + 2e^{-2t} \end{bmatrix}$$

Zero I/P Resp. $= ZIR = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$

Now Zero State Response $(ZSR) = L^{-1}\{\phi(s)\, B\, U(s)\}$ where $U(s) = 1/s$

$$ZSR = L^{-1}\left\{ \begin{bmatrix} \frac{(s+3)}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \end{bmatrix} \right\} = L^{-1} \begin{bmatrix} \frac{1}{s(s+2)(s+1)} \\ \frac{1}{(s+2)(s+1)} \end{bmatrix}$$

$$ZSR = L^{-1} \begin{bmatrix} \frac{1/2}{s} - \frac{1}{(s+1)} + \frac{1/2}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Now $x(t) = ZIR + ZSR = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$

$$x(t) = \begin{bmatrix} 0.5 + 2e^{-t} - 1.5e^{-2t} \\ -2e^{-t} + 3e^{-2t} \end{bmatrix} \quad \text{Ans.}$$

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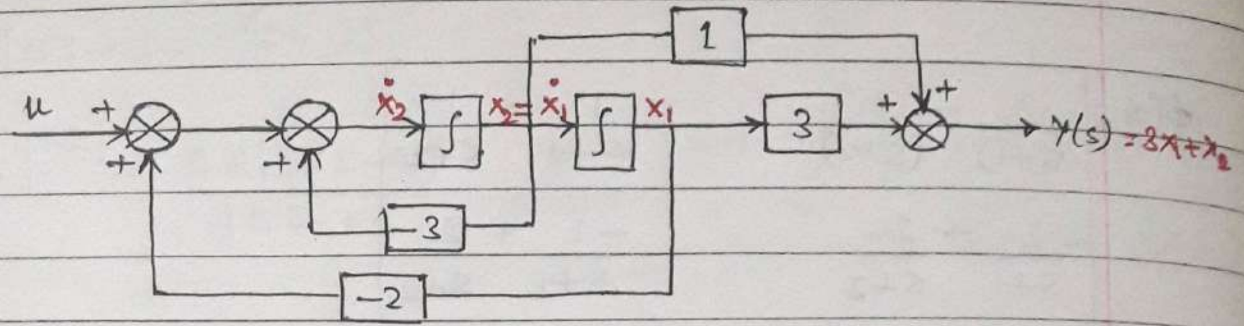
Q8. Construct the state model for $G(s) = \frac{s+3}{s^2+3s+2}$

Sol^n Using direct decomposition, $G(s) = \frac{3}{s^2+3s+2} + \frac{s}{s^2+3s+2}$



From the above state diagram foll'g eq^{n}s are used-

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 3x_2 + u$$

O/P eq^n is $\quad y = 3x_1 + x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{State Model}$$

$$[y] = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore [A] = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 3 & 1 \end{bmatrix}, \; D = [0]$$

Q9. The state eq^n of a linear time invariant system is given as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad \text{for } t > 0$$

Find the controllability and state transition matrix

Sol^n From the given state model;

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } n = 2$$

$$Q_c = [B : AB] \quad \text{where } AB = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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$$\therefore Q_c = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \text{ and } |Q_c| = \begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix} = 0$$

Thus the rank of $Q_c \neq n$, so the system is not state controllable.

Now State Transition Matrix $= \phi(t) = L^{-1}[sI-A]^{-1}$

$$[sI-A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ -1 & s+1 \end{bmatrix}$$

$$\text{Adj}[sI-A] = \begin{bmatrix} s+1 & 1 \\ 0 & s+2 \end{bmatrix}^T = \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix}$$

and $|sI-A| = (s+2)(s+1)$

$$\therefore \phi(s) = [sI-A]^{-1} = \frac{\text{Adj}(sI-A)}{|sI-A|} = \frac{1}{(s+2)(s+1)} \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix}$$

$$\phi(s) = \begin{bmatrix} \frac{1}{(s+2)} & 0 \\ \frac{1}{(s+2)(s+1)} & \frac{1}{s+1} \end{bmatrix}$$

$$\phi(t) = L^{-1}\phi(s) = L^{-1}[sI-A]^{-1} = L^{-1} \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+2)(s+1)} & \frac{1}{s+1} \end{bmatrix}$$

$$L^{-1}\phi(s) = \begin{bmatrix} \frac{1}{(s+2)} & 0 \\ \frac{1}{(s+1)} - \frac{1}{(s+2)} & \frac{1}{(s+1)} \end{bmatrix} \text{ or } \phi(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

Q10. Test the controllability of the system $[\dot{x}] = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u$

Solⁿ— Given, $A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}$

and $n = 3$ Since $Q_c = [B : AB : A^2B]$

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$$AB = \begin{bmatrix} -3 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} -3 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 2 & 4 \\ 2 & 1 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 1 & 0 & -3 & 2 & 9 \\ 0 & 0 & 2 & 0 & 2 & 4 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

Considering the determinant of order 3×3

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 2 \end{vmatrix} = 4 \neq 0$$

Hence the rank of $Q_c = 3 = n$ so it is completely controllable.

Q11. A system is described by the eqn-

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Determine $\phi(t)$ and transfer function of the system.

Soln- from the given model, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$
$D = [0]$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$Adj(sI - A) = \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|sI - A| = s(s+3) + 2 = s^2 + 3s + 2 = (s+1)(s+2)$$

$$\phi(s) = [sI - A]^{-1} = \frac{Adj(sI-A)}{|sI-A|} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\phi(s) = \begin{bmatrix} \frac{2}{(s+1)} - \frac{1}{(s+2)} & \frac{1}{(s+1)} - \frac{1}{(s+2)} \\ \frac{-2}{(s+1)} + \frac{2}{(s+1)} & \frac{-1}{(s+1)} + \frac{2}{(s+2)} \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}\phi(s) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \quad \underline{\text{Ans}}$$

Now $T(s) = C\,[sI-A]^{-1}\,B = C\,[\phi(s)]\,B$

$$T(s) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{1\times2} \underbrace{\begin{bmatrix} \frac{(s+3)}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{2\times1}$$

$$T(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} \end{bmatrix} = \frac{s}{(s+1)(s+2)} = \frac{s}{s^2+3s+2} \quad \text{Ans}$$

Q12. for the system $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$$

Predict the observability for the system.

$\underline{\underline{\text{Sol}^n}}$ – The order of the system is $n = 3$

Observability test matrix, $Q_0 = [C^T : A^TC^T : (A^T)^2C^T]$

$$A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$$

$$\therefore \quad Q_0 = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$|Q_0| = \begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 4[7+5] + 6[-5-5] + 6[-5+7]$$

$$|Q_0| = 48 - 60 + 12 = 0$$

As $|Q_0| = 0$, the rank of $Q_0 = 2 \neq n$ hence the system is not completely observable.

LNCT GROUP OF COLLEGES

Assignment on Unit-V Design of Control Systems

1. What is Proportional controller and what are its advantages?
2. What is the drawback in P-controller?
3. What is integral control action?
4. What is the advantage and disadvantage in integral controller?
5. What is PI controller?
6. What is PD controller?
7. What is PID controller?
8. Discuss lead compensator. Sketch the Bode plot of a lead compensator. Give the design steps of a lead compensator.
9. Discuss lag compensator. Sketch the Bode plot of a lag compensator. Give the design steps of a lag compensator.
10. Sketch the Bode plot and pole-zero plot of a lead compensator.
11. The transfer function of a system is given by $\frac{C(s)}{R(s)} = \frac{s^3+3s+3}{s^3+6s^2+11s+6}$. Obtain the stste model in phase variable form.
12. A feedback system has a closed loop transfer function $\frac{C(s)}{R(s)} = \frac{10(s+4)}{s(s+1)(s+3)}$. Construct the state model using parallel decomposition.
13. The state equation of a linear time- invariant system is given by $\dot{x} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$. Find the state vector x(t) for t≥0 where u(t) is a unit step input.
14. A system is described by $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 2y = 6u$ where y is the output and u is the input of the system. Obtain the state space representation of the system.
15. Obtain the state transition matrix for the system given by $\dot{x} = Ax$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$.

REFERENCES UNIT V:

1) Control System Engineering by Norman S. Nise, John Wiley & Sons, Inc, 6th edition.

2) Modern Control Engineering by K. Ogata, Pearson Education, New Delhi.

3) Linear Control System by B.S. Manke, Khanna Publishers.

4) Feedback and Control Systems by Joseph J. Distefano III , Allen R. Stubberud, Ivano J. Williams (Schaum's ouTlines) TMH, 2nd edition.

5) Automatic Control systems by Benjamin C. Kuo, Wiley India Pvt. Ltd, 9th Edition.

6) Automatic Control Systems by S. Hasan Saeed, S.K. Kataria & Sons, New Delhi.